

**(1a)**

Let

$$z = a + bi$$

$$w = c + di$$

LHS:

$$\begin{aligned} & \overline{z + w} \\ &= \overline{(a + bi) + (c + di)} \\ &= \overline{(a + c) + (b + d)i} \\ &= (a + c) - (b + d)i \end{aligned}$$

RHS:

$$\begin{aligned} & \bar{z} + \bar{w} \\ &= \overline{(a + bi)} + \overline{(c + di)} \\ &= (a - bi) + (c - di) \\ &= (a + c) - (b + d)i \end{aligned}$$

**(1b)**

LHS:

$$\begin{aligned} & \overline{z\bar{w}} \\ &= \overline{(a + bi)(c + di)} \\ &= \overline{ac + adi + bci + bdi^2} \\ &= (ac - bd) - (ad + bc)i \end{aligned}$$

RHS:

$$\begin{aligned} & \bar{z}\bar{w} \\ &= \overline{(a + bi)}\overline{(c + di)} \\ &= (a - bi)(c - di) \\ &= ac - adi - bci + bdi^2 \\ &= (ac - bd) - (ad + bc)i \end{aligned}$$

(1c)

LHS:

$$\begin{aligned}
 & \overline{(z/w)} \\
 = & \overline{\left(\frac{a+bi}{c+di}\right)} \\
 = & \overline{\left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i} \\
 = & \left(\frac{ac+bd}{c^2+d^2}\right) - \left(\frac{bc-ad}{c^2+d^2}\right)i
 \end{aligned}$$

RHS:

$$\begin{aligned}
 & \bar{z}/\bar{w} \\
 = & \overline{(a+bi)}/\overline{(c+di)} \\
 = & \frac{a-bi}{c-di} \\
 = & \left(\frac{ac+bd}{c^2+d^2}\right) - \left(\frac{bc-ad}{c^2+d^2}\right)i
 \end{aligned}$$

(2)

The sum  $\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2}$  can be expressed as

$$\frac{s^2t^2 + r^2t^2 + r^2s^2}{r^2s^2t^2}$$

Here's the tricky part, we try to rewrite the numerator and express the fraction as

$$\begin{aligned}
 & = \frac{(st+rt+rs)^2 - 2(r^2st + rs^2t + rst^2)}{(rst)^2} \\
 & = \frac{(st+rt+rs)^2 - 2(rst)(r+s+t)}{(rst)^2}
 \end{aligned}$$

From the polynomial  $x^3 - 6x^2 + 5x - 7 = 0$  we can find the following with Vieta's formulas

$$\begin{aligned} r + s + t &= 6 \\ rs + st + tr &= -5 \\ rst &= 7 \end{aligned}$$

Now the desired sum  $\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2}$  is

$$\begin{aligned} &= \frac{(-5)^2 - 2(7)(6)}{7^2} \\ &= -\frac{59}{49} \end{aligned}$$

Our answer is just  $\boxed{-\frac{59}{49}}$

(3)

The roots are just the 8 8th of unity which form an octagon in the complex plane. The roots with positive part are:

$$\boxed{\sqrt{2} + \sqrt{2}i, 1, \sqrt{2} - \sqrt{2}i}$$

(4)

The polynomial  $x^4 + x^2 + 1 = 0$  can be rewritten as

$$\frac{x^6 - 1}{x^2 - 1} = 0$$

(recall the formula for the sum of a geometric series)

The zeros of  $x^6 - 1$  is just the 6 6th roots of unity. But we cannot have 6 roots since the original polynomial is of degree 4. Why do we have 2 extra roots? It's because the denominator is 0 at  $x = 1$  and  $x = -1$ . Our desired roots are just:

$$\boxed{\frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i}$$

(5)

We are given that  $(a + bi)z$  is equidistant from the origin and  $z$ . This translates to

$$\begin{aligned} |(a + bi)z - z| &= |(a + bi)z| \\ |z(a - 1) + bzi| &= |az + bzi| \\ |z|(a - 1) + bi| &= |z||a + bi| \\ (a - 1)^2 + b^2 &= a^2 + b^2 \end{aligned}$$

This gives

$$a = \frac{1}{2}$$

And it follows that

$$b = \frac{255}{4}$$

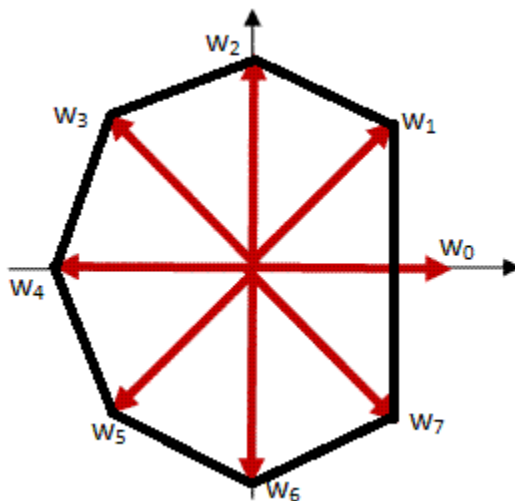
(c1)

First we rewrite the sum as

$$\frac{x^8 - 1}{x - 1} = 0$$

This is just the 8 8th roots of unity, but  $x \neq 1$ .

A sketch of the region



The area can be computed with simple geometry to get  $A = \frac{2+\sqrt{2}}{4}$

(c2)

Suppose the four roots of  $f(x) = x^4 - 3x^2 + x - 9 = 0$  are  $x_1, x_2, x_3,$  and  $x_4$ .

Their reciprocals are  $\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3},$  and  $\frac{1}{x_4}$ .

These reciprocals satisfy  $\frac{1}{x^4} - \frac{3}{x^2} + \frac{1}{x} - 9 = 0$

But this is not a polynomial, so we multiply by  $x^4$

$$g(x) = x^4 f\left(\frac{1}{x}\right) = 1 - 3x + x^3 - 9x^4$$

(c3)

We know from Euler's formula that

$$e^{ix} = \cos(x) + i \sin(x)$$

Suppose we take the imaginary part of both sides, then

$$\text{Im}(e^{ix}) = \text{Im}(\cos(x) + i \sin(x)) = \sin(x)$$

We can rewrite the integral as

$$\begin{aligned} & \int x e^x \text{Im}(e^{ix}) dx \\ &= \text{Im}\left(\int x e^x e^{ix} dx\right) \\ &= \text{Im}\left(\int x e^{(1+i)x} dx\right) \end{aligned}$$

Integration by parts...

$x$	$e^{(1+i)x}$
$1$	$\frac{e^{(1+i)x}}{1+i}$
$0$	$\frac{e^{(1+i)x}}{(1+i)^2}$

Yields

$$\begin{aligned}
&= \operatorname{Im} \left( \frac{x e^{(1+i)x}}{1+i} - \frac{e^{(1+i)x}}{(1+i)^2} \right) \\
&= \operatorname{Im} \left( \frac{x e^x e^{ix} (1-i)}{2} - \frac{e^x e^{ix}}{2i} \right) \\
&= \operatorname{Im} \left( \frac{x e^x e^{ix} (1-i)}{2} + \frac{i e^x e^{ix}}{2} \right) \\
&= \operatorname{Im} \left( \frac{x e^x (\cos(x) + i \sin(x)) (1-i)}{2} + \frac{i e^x (\cos(x) + i \sin(x))}{2} \right) \\
&= \operatorname{Im} \left( \frac{x e^x}{2} (\cos(x) - i \cos(x) + i \sin(x) + \sin(x)) + \frac{e^x}{2} (i \cos(x) - \sin(x)) \right) \\
&= \frac{e^x}{2} (x \sin(x) - x \cos(x) + \cos(x)) + c \\
&\quad \boxed{\frac{e^x}{2} (x \sin(x) - x \cos(x) + \cos(x)) + c}
\end{aligned}$$